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Exponential Random Energy Model

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Abstract

In this paper the Random Energy Model(REM) under exponential type environment is considered which includes double exponential and Gaussian cases. Limiting Free Energy is evaluated in these models. Limiting Gibbs' distribution is evaluated in the double exponential case.

Key words: Spin Glasses; Random Energy Model; Double Exponential; Free Energy; Gibbs' Distribution.

1 Introduction

Usually the Random Energy Model[4] is considered in a Gaussian[1, 2, 5, 6, 8] environment. In this paper we discuss the same under a double exponential environment. It is interesting to note that in our analysis the distribution of Hamiltonian, H_N does not depend on N. We use large deviation method[5] to calculate the limiting free energy. There is a phase transition at $\beta = 1$. The methods carry over to a more general exponential type family that includes the Gaussian case as well and we provide explicit formulae for the free energy.

We use Talagrand's[8] approach to obtain the limiting Gibbs' distribution in the low temperature regime. It is interesting to note that the limit is again a Poisson-Dirichlet distribution. Observe that when X is double exponential with parameter one, $\mathbf{E}(e^{\beta X})$ does not exist for $\beta > 1$. For $0 < \beta < 1$ we obtain − as expected − uniform distribution as the infinite

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volume limit of Gibbs' distributions. This is done via an interesting variant of the strong law of large numbers. These methods carry over to the Gaussian case as well.

2 Free energy

For each configuration $\sigma \in \Sigma_N = \{-1,1\}^N$ of an N particle system, the Hamiltonian is $H_N(\sigma)$. Since we are considering REM, $\{H_N(\sigma)\}$ are i.i.d. In fact, we assume that they are double exponential, that is, have density (not depending on N)

$$\phi_N(x) = \frac{1}{2}e^{-|x|}, -\infty < x < \infty.$$

The partition function of the system is

$$Z_N(\beta) = \sum_\sigma e^{-\beta H_N(\sigma)} = 2^N \mathbf{E}_\sigma e^{-N\beta \frac{H_N(\sigma)}{N}},$$

where $\beta > 0$ is the inverse temperature and $\mathbf{E}_\sigma$ stands for expectation w.r.t σ when Σ_N has uniform distribution. Hence the free energy of the system is

$$\frac{1}{N}\log Z_N(\beta) = \log 2 + \frac{1}{N}\log \mathbf{E}_\sigma e^{-N\beta\frac{H_N(\sigma)}{N}}.$$

Now let μ_N be the induced (random) probability on $\mathbf{R}$ via the map

$$\sigma \mapsto \frac{H_N(\sigma)}{N}$$

when Σ_N has uniform distribution.

Proposition 2.1 *$\mu_N \Rightarrow \delta_0$ a.s. as $N \to \infty$.*

That is, for almost every sample point, the sequence of random measures $\{\mu_N\}$ converges weakly to point mass at 0.

Proof: For any $\epsilon > 0$, define $\triangle(\epsilon) = [-\epsilon, \epsilon] \subset \mathbf{R}$. Now by Markov inequality,

$$\mathbf{P}(\mu_N(\triangle^c(\epsilon)) > \epsilon) < \frac{1}{\epsilon}\mathbf{E}\mu_N(\triangle^c(\epsilon)) < \frac{1}{\epsilon}\mathbf{P}(|H_N| > \epsilon N) = \frac{1}{\epsilon}e^{-\epsilon N}.$$

Apply Borel-Cantelli.

Let $\triangle \subseteq \mathbf{R}$ be an open interval. Put $m = \inf_{x \in \triangle} |x|$ and $M = \sup_{x \in \triangle} |x|$, and $q_N = P(\frac{H_N}{N} \in \triangle)$. These quantities, of course, depend on $\triangle$. Observe that $\triangle \subseteq (-M, -m] \cup [m, M)$, so that

$$q_N \leq \int_{Nm}^{NM} e^{-x} dx \leq \int_{Nm}^{\infty} e^{-x} dx = e^{-Nm} \tag{1}$$

and

$$q_N \geq \frac{1}{2} \int_{Nm}^{NM} e^{-x} dx > \frac{1}{2} \int_{Nm}^{Nm+\delta} e^{-x} dx > \frac{\delta}{2} e^{-(Nm+\delta)}, \tag{2}$$

for any δ, $(0 < \delta < M - m)$. Both (1) and (2) remain true even if $m = 0$.

Proposition 2.2 *If $m > \log 2$, then almost surely eventually $\mu_N(\triangle) = 0$.*
Hence almost surely eventually, $\mu_N[-\log 2, \log 2] = 1$

Proof: By definition,

$$\mu_N(\triangle) = \frac{1}{2^N} \sum_{\sigma} \mathbf{1}_{\frac{H_N(\sigma)}{N}}(\triangle)$$

Hence $\{\mu_N(\triangle) > 0\} = \{\sum_{\sigma} \mathbf{1}_{\frac{H_N(\sigma)}{N}}(\triangle) > 1\}$. Now by Chebyscheff's inequality, $\mathbf{P}(\mu_N(\triangle) > 0) < \mathbf{E} \sum_{\sigma} \mathbf{1}_{\frac{H_N(\sigma)}{N}}(\triangle) = 2^N q_N$.

Since $m > \log 2$, (1) implies, $\sum_{N \geq 1} \mathbf{P}(\mu_N(\triangle) > 0) < \infty$. Borel-Cantelli completes the proof.

Proposition 2.3 *If $m < \log 2$, then for any $\epsilon > 0$ a.s. eventually*

$$(1 - \epsilon) q_N \leq \mu_N(\triangle) \leq (1 + \epsilon) q_N.$$

Proof: Note that

$$\begin{aligned}
var(\mu_N(\triangle)) &= \mathbf{E}(\mu_N(\triangle))^2 - [\mathbf{E}\mu_N(\triangle)]^2 \\
&= \tfrac{1}{2^{2N}} \sum_{\sigma,\tau} \left[\mathbf{E} \mathbf{1}_{\frac{H_N(\sigma)}{N}}(\triangle) \mathbf{1}_{\frac{H_N(\tau)}{N}}(\triangle) - q_N^2 \right] \\
&= \tfrac{1}{2^{2N}} \sum_{\sigma=\tau} \left[\mathbf{E} \mathbf{1}_{\frac{H_N(\sigma)}{N}}(\triangle) \mathbf{1}_{\frac{H_N(\tau)}{N}}(\triangle) - q_N^2 \right] \\
&\leq \tfrac{1}{2^{2N}} \sum_{\sigma} \mathbf{E} \mathbf{1}_{\frac{H_N(\sigma)}{N}}(\triangle) \\
&= \tfrac{q_N}{2^N}
\end{aligned}$$

Hence for any $\epsilon > 0$, by Chebycheff's inequality

$$\mathbf{P}\left[|\mu_N(\triangle) - \mathbf{E}\mu_N(\triangle)| > \epsilon \mathbf{E}\mu_N(\triangle)\right] < \frac{1}{\epsilon^2 2^N q_N}.$$

Using (2) and the fact $m < \log 2$, we get $\sum_N 2^{-N} q_N^{-1} < \infty$, so that $\sum_N \mathbf{P}(|\mu_N(\triangle) - q_N| > \epsilon q_N) < \infty$. Borel-Cantelli completes the proof.

Propositions 2.3 and 2.2 combined with the inequalities (1) and (2) yield,

Proposition 2.4 *Almost surely,*

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N} \log \mu_N(\triangle) &= -m \quad \textit{if } m < \log 2 \\ &= -\infty \quad \textit{if } m > log 2. \end{aligned}$$

Now let us consider the map $I : \mathbf{R} \to \mathbf{R}^+$, defined as follows,

$$\begin{aligned} I(x) &= |x| \quad \text{if } -\log 2 \le x \le \log 2 \\ &= \infty \quad \text{otherwise.} \end{aligned}$$

Theorem 2.1 *Almost surely, the sequence $\{\mu_N\}$ satisfies the large deviation principle with rate function I.*

Proof: The collection $\mathcal{A}$ of open intervals with rational end points is a countable base for $\mathbf{R}$. For $\triangle \in \mathcal{A}$, put $L_\triangle = -\lim_{N\to\infty} \frac{1}{N} \log \mu_N(\triangle)$. Note that, for $x \in \mathbf{R}$ Proposition 2.4 implies, $I(x) = \sup_{x \in \triangle \in \mathcal{A}} L_\triangle$.

Since almost surely $\{\mu_N\}$ is supported on a compact set, Theorem 4.1.11 of Dembo and Zeitouni[3] completes the proof.

Theorem 2.2 *For almost every sample point,*

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N} \log Z_N(\beta) &= \log 2 \quad \textit{if } \beta \le 1 \\ &= \beta \log 2 \quad \textit{if } \beta \ge 1. \end{aligned}$$

Proof: By Theorem 2.1, almost surely, the sequence $\{\mu_N\}$ satisfies large deviation principle with rate function I. By Proposition 2.2, the sequence $\{\mu_N\}$ is supported on a compact set. Varadhan's lemma with $h(x) = \beta x$, $-\infty < x < \infty$ gives

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N} \log Z_N(\beta) &= \log 2 - \inf_{x \in \mathbf{R}} \{h(x) + I(x)\} \\ &= \log 2 - \inf_{|x| \le \log 2} \{\beta x + |x|\} \\ &= \log 2 - \inf_{0 \le x \le \log 2} (1-\beta) x. \end{aligned}$$

Hence a.s.

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N} \log Z_N(\beta) &= \log 2 \qquad \text{if } \beta \le 1 \\ &= \beta \log 2 \quad \text{if } \beta \ge 1 \end{aligned}$$

Remark 2.1 It is worth noting that the above consideration hold for a general class of distributions. More precisely, let $\alpha \ge 1$ be fixed. For each N, consider $\{H_N(\sigma), \sigma \in \Sigma\}$ to be i.i.d. with density

$$\phi_N(x) = C_{\alpha,N}\, e^{-\frac{|x|^\alpha}{\alpha N^{\alpha-1}}}, \quad -\infty < x < \infty,$$

where $C_{\alpha,N} = \frac{1}{2\Gamma(\frac{1}{\alpha})} \left(\frac{\alpha}{N}\right)^{\frac{\alpha-1}{\alpha}}$.

Of course, when $N = 1$, this reduces to the case considered above and for $N = 2$ this becomes the Gaussian case usually considered in the literature. For $\alpha > 1$, similar calculations as above lead to the rate function

$$\begin{aligned} I(x) &= \tfrac{|x|^\alpha}{\alpha} \quad \text{if } -(\alpha \log 2)^{\frac{1}{\alpha}} \le x \le (\alpha \log 2)^{\frac{1}{\alpha}} \\ &= \infty \quad\ \ \text{otherwise.} \end{aligned}$$

and almost surely, the limiting free energy is

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N} \log Z_N(\beta) &= \log 2 + \tfrac{\alpha-1}{\alpha} \beta^{\frac{\alpha}{\alpha-1}} \quad \text{if } \beta \le (\alpha \log 2)^{\frac{\alpha-1}{\alpha}} \\ &= \beta(\alpha \log 2)^{\frac{1}{\alpha}} \qquad\quad \text{if } \beta > (\alpha \log 2)^{\frac{\alpha-1}{\alpha}} \end{aligned}.$$

For $\alpha = 2$, this coincides with the known formula[8]. Of course, when $\alpha = 1$ the formula, interpreted in the limiting sense, is the one obtained earlier.

3 Gibbs' Distribution

We return to the double exponential environment. Recall that Gibbs' distribution for the N particle system is the (random) probability on Σ_N defined as

$$G_N(\sigma) = \frac{e^{-\beta H_N(\sigma)}}{Z_N(\beta)}, \quad \sigma \in \Sigma_N.$$

We show that for $\beta < 1$, the (random) Gibbs' distribution G_N converges weakly to the uniform probability on $\{-1,1\}^\infty$ almost surely. Uniform probability here means the product probability on $\{-1,1\}^\infty$ where each coordinate space has $(\frac{1}{2}, \frac{1}{2})$ probability. Since for each N, G_N is defined on $\{-1,1\}^N$ the notion of convergence here is to be carefully understood. This is made precise in Theorem 3.1 below.

Theorem 3.1 *Fix $\beta < 1$. Then almost surely, for any $K \geq 1$ and any $\sigma \in \{-1,1\}^K$, $\rho_N(\sigma) \to \frac{1}{2^K}$ as $N \to \infty$, where ρ_N is the marginal of G_N on $\{-1,1\}^K$.*

Proof: For $0 < \beta < 1$, define $Z'_N(\beta) = \sum_\sigma e^{-\beta H_N(\sigma)} \mathbf{1}_{\{-H_N(\sigma) \leq \delta N\}}$, where δ will be chosen latter depending on β.

Since $\mathbf{P}\{-H_N(\sigma) > \delta N\} \leq \frac{1}{2} e^{-\delta N}$, for any $\delta > log 2$, Borel-Cantelli implies that almost surely, eventually

$$Z_N(\beta) = Z'_N(\beta). \tag{3}$$

Argument as in Proposition 2.3 and symmetry of distribution of H_N lead to,

$$\mathbf{P}\left[|Z'_N(\beta) - \mathbf{E}Z'_N(\beta)| > \epsilon \mathbf{E}Z'_N(\beta)\right] < \frac{\mathbf{E}e^{2\beta H_N} \mathbf{1}_{\{H_N \leq \delta N\}}}{\epsilon^2 2^N (\mathbf{E}e^{\beta H_N} \mathbf{1}_{\{H_N \leq \delta N\}})^2}. \tag{4}$$

But,

$$\mathbf{E}e^{\beta H_N} \mathbf{1}_{\{H_N \leq \delta N\}} > \frac{1}{1+\beta}. \tag{5}$$

Now note that,

$$\begin{array}{lcll} \mathbf{E}e^{2\beta H_N} \mathbf{1}_{\{H_N \leq \delta N\}} & \leq & \frac{1}{1-4\beta^2} & \text{if } \beta < \frac{1}{2} \\ & = & \frac{1+\delta N}{2} & \text{if } \beta = \frac{1}{2} \\ & \leq & \frac{1}{2(2\beta-1)} e^{(2\beta-1)\delta N} & \text{if } \beta > \frac{1}{2}. \end{array} \tag{6}$$

In case $0 < \beta \leq \frac{1}{2}$, we choose $\delta > \log 2$ while for $\frac{1}{2} < \beta < 1$ we choose δ, $\log 2 < \delta < \frac{\log 2}{2\beta - 1}$ so that by (5) and (6), (4) implies

$$\sum_{N \geq 1} \mathbf{P}\left[|Z'_N(\beta) - \mathbf{E}Z'_N(\beta)| > \epsilon \mathbf{E}Z'_N(\beta)\right] < \infty.$$

Thus, with the choice of δ as specified above, Borel-Cantelli implies that almost surely eventually,

$$(1-\epsilon)\mathbf{E}Z'_N(\beta) \leq Z'_N(\beta) \leq (1+\epsilon)\mathbf{E}Z'_N(\beta).$$

Combining this with (3) we have almost surely eventually,

$$(1-\epsilon)\mathbf{E}e^{2\beta H_N} \mathbf{1}_{\{H_N \leq \delta N\}} \leq \frac{Z_N(\beta)}{2^N} \leq (1+\epsilon)\mathbf{E}e^{2\beta H_N} \mathbf{1}_{\{H_N \leq \delta N\}}.$$

Now fix $K \geq 1$ and $\sigma \in \{-1,1\}^K$. Let $Y_N = \displaystyle\sum_{\sigma' \succ \sigma} e^{-\beta H_N(\sigma')}$, where the sum is over all $\sigma' \in \{-1,1\}^N$ that extend σ.

Argument similar to above shows that, with the same δ, almost surely eventually,

$$(1-\epsilon)\mathbf{E}e^{2\beta H_N}\mathbf{1}_{\{H_N\le\delta N\}} \le \frac{Y_N}{2^{N-K}} \le (1+\epsilon)\mathbf{E}e^{2\beta H_N}\mathbf{1}_{\{H_N\le\delta N\}}.$$

As a consequence, almost surely eventually $\rho_N(\sigma)$, which by definition is $\frac{Y_N}{Z_N(\beta)}$, lies between $\frac{1-\epsilon}{1+\epsilon}\,2^{-K}$ and $\frac{1+\epsilon}{1-\epsilon}\,2^{-K}$ completing the proof.

Remark 3.1 Clearly for the regime $\beta < 1$ the argument shows directly that almost surely $\frac{1}{N}\log Z_N(\beta) \to \log 2$.

Remark 3.2 Returning to Remark 2.1, if we consider the environment parametrized by α, it is in general difficult to evaluate the limiting Gibbs' distribution. However for $\alpha = 2$, our arguments lead to the convergence of Gibbs' distribution to the uniform probability, in high temperature regime. (See appendix for details.)

Remark 3.2 Hidden in the above argument is a variant of the strong law of large numbers, which will be taken up elsewhere.

To study the Gibbs' distribution for $\beta > 1$, since multiplicative constant cancels out, instead of $H_N(\sigma)$ we use the random variables $H'_N(\sigma) = H_N(\sigma) + a_N$, where $a_N = (N-1)\log 2$. Mimicking the proof of Lemma 1.2.2 of Talagrand[8] yields,

Proposition 3.1 *For $b \in \mathbf{R}$,*

$$\lim_{N\to\infty} \mathbf{P}(\#\{\sigma : -H'_N(\sigma) \ge b\} = k) = e^{-e^{-b}}\frac{e^{-kb}}{k!}.$$

Moreover, if $\exists$ exactly k many $\sigma^1, \cdots, \sigma^k$ in Σ_N such that $-H'_N(\sigma^i) \ge b$ then for large N these k points are distributed like $\{X_1, \cdots, X_k\}$ where X_i's are i.i.d. with density $e^{-(t-b)}\mathbf{1}_{\{t\ge b\}}$.

Proof: For fixed $b \in \mathbf{R}$ and N so large that $b + a_N > 0$, define

$$d_N(b) = \mathbf{P}(-H'_N(\sigma) \ge b) = \frac{1}{2}\int_b^\infty e^{-(x+a_N)}dx = \frac{1}{2}e^{-(b+a_N)}. \qquad (7)$$

By definition of a_N clearly, $2^N d_N(b) = e^{-b}$. Since $\#\{\sigma : -H'_N(\sigma) \ge b\}$ is Binomial with parameters 2^N and d_N, Poisson approximation of the binomial completes proof of the first part.

The last part of the proposition follows from the fact that H'_N's are i.i.d. and by (7), have density proportional to $e^{-(t+a_N)}\mathbf{1}_{\{t\ge b\}}$.

Let $m > 0$ and Π be the Poisson point process on $(0, \infty)$ with intensity $x^{-m-1}dx$. Then almost surely, Π consists of summable sequences and hence can be arranged in decreasing order $(\pi(1), \pi(2), \cdots)$. Let $S(\pi)$ denotes the sum $\sum \pi(i)$. The distribution of $(\frac{\pi(i)}{S(\pi)} : i \geq 1)$ is denoted by $PD(m, 0)$, called Poisson-Dirichlet distribution with parameter m. For more details and a two parameters family see Pitman and Yor[7].

Proposition 3.2 *Consider a Poisson point process on* $\mathbf{R}$ *with intensity* $e^{-x}dx$, *and* $(c_i)_{i\geq 1}$ *an enumeration in decreasing order of these Poisson points. Then the sequence*

$$v_i = \frac{e^{\beta c_i}}{\sum_j e^{\beta c_j}}$$

has distribution $PD(\frac{1}{\beta}, 0)$.

Proof: If (c_i) are Poisson points on $\mathbf{R}$ with intensity $e^{-x}dx$, $u_i = Ke^{\beta c_i}$, where $K = \beta^\beta$, then (u_i) are Poisson points on $(0, \infty)$ with intensity $x^{-\frac{1}{\beta}-1}dx$. Clearly if (c_i) are enumerated in decreasing order then so are (u_i). Since $v_i = u_i / \sum_j u_j$, we conclude that (v_i) follows $PD(\frac{1}{\beta}, 0)$.

Let $\mathcal{S}$ be the set of all decreasing non-negative sequences with sum at most one. With the l^1-metric $d(\tilde{x}, \tilde{y}) = \sum |x_i - y_i|$, where $\tilde{x} = (x_i)$ and $\tilde{y} = (y_i)$, $\mathcal{S}$ is a Polish space. Now let $\tilde{w} = (w_i)_{i\geq 1}$, where $(w_i)_{i\leq 2^N}$ is the non-increasing enumeration of the (random) Gibbs' weights $\{G_N(\sigma) : \sigma \in \Sigma_N\}$ with $w_i = 0$ for $i > 2^N$. Let μ_N be the law of $\tilde{w}$ on $\mathcal{S}$. Let μ be the law of $\tilde{v} = (v_i)_{i\geq 1}$ of Proposition 3.2 on $\mathcal{S}$.

Theorem 3.2 *Let* $\beta > 1$. *Then* $\mu_N \Rightarrow \mu$ *on* $\mathcal{S}$, *that is, the law of* $\tilde{w}$ *converges to* $PD(\frac{1}{\beta}, 0)$ *as* $N \to \infty$.

Proof: One has only to adapt the proof of Theorem 1.2.1 in Talagrand[8]. At the suggestion of the referee we give a brief outline. Fix f uniformly continuous function on $\mathcal{S}$, bounded by one and $\epsilon > 0$. Suffices to show $|\int f d\mu_N - \int f d\mu| < \epsilon$ for all large N.

Let (c_i) be as in Proposition 3.2. Recall that $a_N = (N-1)\log 2$ and $H'_N(\sigma) = H_N(\sigma) + a_N$. For fixed $b \in \mathbf{R}$, Put temporarily,

$$\begin{array}{lcl}
Z_N(\beta) & = & \sum_\sigma e^{-\beta H'_N(\sigma)}, \\
Z_N(\beta, b) & = & \sum_\sigma e^{-\beta H'_N(\sigma)} \mathbf{1}_{\{-H'_N(\sigma)\geq b\}}, \\
w_i^b & = & \frac{e^{\beta h_i}}{Z_N(\beta,b)} \mathbf{1}_{\{h_i\geq b\}} \text{ if } w_i = \frac{e^{\beta h_i}}{Z_N(\beta)}, \\
v_i^b & = & \frac{e^{\beta c_i}\mathbf{1}_{\{c_i\geq b\}}}{\sum e^{\beta c_i}\mathbf{1}_{\{c_i\geq b\}}}.
\end{array}$$

Denote $\tilde{w}^b = (w_i^b)_{i\geq 1}$ and $\tilde{v}^b = (v_i^b)_{i\geq 1}$. Let μ_N^b (respectively, μ^b) be the law of $\tilde{w}^b$ (respectively, $\tilde{v}^b$) on $\mathcal{S}$. Observe,

$$\begin{aligned} |\textstyle\int f d\mu_N - \int f d\mu| \quad < \quad & |\textstyle\int f d\mu_N - \int f d\mu_N^b| + |\int f d\mu_N^b - \int f d\mu^b| \\ & +|\textstyle\int f d\mu^b - \int f d\mu|. \end{aligned} \tag{8}$$

Firstly, we show that for given $\delta > 0$, there exists b_0 (depending on β and δ) such that for $b \leq b_0$,

$$\limsup_{N\to\infty} \mathbf{P}\left(\frac{Z_N(\beta) - Z_N(\beta, b)}{Z_N(\beta)} \geq \delta\right) \leq \delta. \tag{9}$$

Then, using $d(\tilde{w}, \tilde{w}^b) = 2\frac{Z_N(\beta)-Z_N(\beta,b)}{Z_N(\beta)}$ and that f is bounded and uniformly continuous, the first term on the right side of (8) can be made $< \frac{\epsilon}{3}$ for all large N. To see (9), we proceed as follows:

Since $Z_N(\beta) \leq e^{\beta x} \Rightarrow \{\#\{\sigma : -H_N(\sigma) \geq x\} = 0\}$, by Proposition 3.1, $\lim\limits_{N\to\infty} \mathbf{P}(\#\{\sigma : -H_N(\sigma) \geq x\} = 0) = e^{-e^{-x}}$, so there exists $\eta > 0$ such that for large N,

$$\mathbf{P}(Z_N(\beta) \leq \eta) \leq \frac{\delta}{2}. \tag{10}$$

Again, for fixed $x \in \mathbf{R}$ with N so large that $x + a_N > 0$, $\mathbf{E}(Z_N(\beta) - Z_N(\beta, x)) = \frac{1}{\beta-1}e^{(\beta-1)x} - \frac{2}{\beta^2-1}e^{-(\beta-1)a_N}$. Since $\beta > 1$ this can be made sufficiently small by an appropriate choice of large negative quantity x and large N. So that, by Chebycheff's inequality, we can get b_0 (depending on η, δ, β) such that for $b \leq b_0$ and large N,

$$\mathbf{P}[Z_N(\beta) - Z_N(\beta, b) \geq \eta\delta] \leq \frac{\delta}{2}. \tag{11}$$

Now (10) and (11) imply (9).

Secondly, the last term in (8) can be made small by choosing b large negative quantity since $\mu^b \Rightarrow \mu$ as $b \to -\infty$. Fix now such a number b. There is no loss to assume (9) also holds.

Finally, the middle term in (8) can be made arbitrary small by choosing N large, since by the last part of Proposition 3.1 $\mu_N^b \Rightarrow \mu^b$.

Thus for large N each of the three terms on the right hand side of (8) can be made smaller than $\frac{\epsilon}{3}$.

Appendix

Proposition: *Consider REM with $H_N \sim$ Gaussian $(0, N)$ and fix $0 < \beta < \sqrt{2\log 2}$. Then almost surely, for any $K \geq 1$ and any $\sigma \in \{-1, 1\}^K$, $\rho_N(\sigma) \to \frac{1}{2^K}$ as $N \to \infty$, where ρ_N is the marginal of G_N on $\{-1, 1\}^K$.*

A stronger version with a difficult proof is in Talagrand[8]. The purpose of this appendix is to explain how the techniques used in Theorem 3.1 above apply to this case.

For $0 < \beta < \sqrt{2\log 2}$, define $Z'_N(\beta) = \sum_\sigma e^{-\beta H_N(\sigma)} \mathbf{1}_{\{-H_N(\sigma)\le \delta N\}}$, where δ will be chosen latter depending on β.

Since $\mathbf{P}\{-H_N(\sigma) > \delta N\} \le \frac{1}{\sqrt{2\pi N}} e^{-\frac{\delta^2 N}{2}}$, for any $\delta > \sqrt{2\log 2}$, Borel-Cantelli implies that almost surely, eventually

$$Z_N(\beta) = Z'_N(\beta). \tag{12}$$

Just as in Theorem 3.1, we have,

$$\mathbf{P}\left[|Z'_N(\beta) - \mathbf{E}Z'_N(\beta)| > \epsilon \mathbf{E}Z'_N(\beta)\right] < \frac{\mathbf{E}e^{2\beta H_N}\mathbf{1}_{\{H_N\le\delta N\}}}{\epsilon^2 2^N (\mathbf{E}e^{\beta H_N}\mathbf{1}_{\{H_N\le\delta N\}})^2}. \tag{13}$$

Since $\delta > \sqrt{2\log 2} > \beta$,

$$\mathbf{E}e^{\beta H_N}\mathbf{1}_{\{H_N\le\delta N\}} = \frac{e^{\frac{\beta^2 N}{2}}}{\sqrt{2\pi N}} \int_{-\infty}^{\delta N} e^{\frac{(x-\beta N)^2}{2N}} dx > \frac{1}{2} e^{\frac{\beta^2 N}{2}}. \tag{14}$$

Now note that,

$$\mathbf{E}e^{2\beta H_N}\mathbf{1}_{\{H_N\le\delta N\}} = \frac{e^{2\beta^2 N}}{\sqrt{2\pi N}} \int_{(2\beta-\delta)N}^{\infty} e^{-\frac{x^2}{2N}} dx.$$

Thus,

$$\begin{array}{rcll} \mathbf{E}e^{2\beta H_N}\mathbf{1}_{\{H_N\le\delta N\}} & \le & e^{2\beta^2 N} & \text{if } \beta \le \frac{\delta}{2} \\ & \le & \frac{1}{(2\beta-\delta)\sqrt{2\pi N}} e^{(2\delta\beta - \frac{\delta^2}{2})N} & \text{if } \beta > \frac{\delta}{2}. \end{array} \tag{15}$$

In case $0 < \beta < \sqrt{\log 2}$, we choose $\delta = 2\sqrt{\log 2}$ so that $\beta \le \frac{\delta}{2}$ and hence by (14) and (15), (13) implies

$$\sum_{N\ge 1} \mathbf{P}\left[|Z'_N(\beta) - \mathbf{E}Z'_N(\beta)| > \epsilon \mathbf{E}Z'_N(\beta)\right] < \frac{4}{\epsilon^2} \sum_{N\ge 1} e^{-N(\log 2 - \beta^2)} < \infty.$$

In case $\sqrt{\log 2} \le \beta < \sqrt{2\log 2}$, one observes that $\sqrt{2\log 2} < 2\beta - \sqrt{2(\beta^2 - \log 2)}$ so that we can choose $\delta \in (\sqrt{2\log 2}, 2\beta - \sqrt{2(\beta^2-\log 2)})$. With such a choice, $\beta > \frac{\delta}{2}$ and again by (14) and (15), (13) implies

$$\sum_{N\ge 1} \mathbf{P}\left[|Z'_N(\beta) - \mathbf{E}Z'_N(\beta)| > \epsilon \mathbf{E}Z'_N(\beta)\right] < \infty.$$

Thus in either case, if δ is chosen as specified, Borel-Cantelli implies that almost surely eventually,

$$(1-\epsilon)\mathbf{E}Z'_N(\beta) \le Z'_N(\beta) \le (1+\epsilon)\mathbf{E}Z'_N(\beta)$$

and the proof is completed by repeating the same argument of Theorem 3.1.

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